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Combined Forced and Free Convection in a Reverse Osmosis System

The influence of combined forced and free convection on the performance of a reverse osmosis system in a horizontal circular pipe is examined. The free convective motion, which is superimposed upon the main axial flow, is caused by buoyancy forces arising from the buildup of a dense solute boundary layer near the membrane surface. The three-dimensional convective diffusion problem is solved by dividing it into a perturbation part accounting for the buoyancy effects present for $Ra \neq 0$ and a nonperturbation part for the intrinsic convective flow pattern present even when $Ra = 0$.

An approximate solution to the nonperturbation equations is obtained from the literature, and the perturbation equations are solved using a stream function-vorticity scheme valid for high Schmidt numbers. The effects of rejection parameter, Rayleigh number, and pressure parameter on the Sherwood number and concentration polarization are studied. Correlations are developed for the asymptotic Sherwood number and the effective axial length at which free convection becomes significant. The numerical results are in reasonable agreement with limiting analytical solutions and with the experimental asymptotic Sherwood numbers measured by Derzansky and Gill (1974) and Hsieh et al. (1976).

C. Y. CHANG
and
J. A. GUIN
Department of Chemical Engineering
Auburn University
Auburn, Alabama 36830

SCOPE

Reverse osmosis, or hyperfiltration as it is sometimes called, is a separation process dependent upon the ability of a semipermeable membrane to selectively retard a solute while passing the solvent. The best known application of this technology is to water desalination, although, as noted by Derzansky and Gill (1974), there are additional important applications.

In the process, the retarded solute gradually tends to

C. Y. Chang is presently with the Metals Development Section, Iowa State University, Ames, Iowa 50011.

build up at the membrane surface in a concentration boundary layer. This phenomenon is referred to as concentration polarization and is deleterious to membrane performance. The increased solute concentration causes an increase in the osmotic pressure at the membrane surface and hence lessens the effective driving force for permeation through the membrane, thus slowing membrane performance.

Because the solute concentration boundary layer is more dense than the bulk solution, the buoyancy forces present in a horizontal tubular reverse osmosis system cause a secondary free convective motion to develop which is superimposed upon the main axial flow. This buoyancy induced convection has been noticed in the experiments of several investigators, including those of Derzansky and Gill (1974). Furthermore, the complex interrelations between the mass transfer and hydrodynamics of the three-dimensional flow make empirical correlations of data difficult to extrapolate confidently. Thus, the objective of this work is to investigate, in a theoretical manner, the reverse osmosis process in the combined forced and free convective flow regime. The resulting correlations will hopefully prove useful to the process design and engineering analysis of reverse osmosis systems.

CONCLUSIONS AND SIGNIFICANCE

Theoretical study of a laminar flow horizontal tubular reverse osmosis system has shown that whenever the product of the dimensionless permeation velocity and the Rayleigh number exceeds about 7×10^3, that is, for $u_w Ra > 7 \times 10^3$, free convection becomes a significant transport mechanism. In this regime, the mass transfer rate and membrane performance are substantially enhanced by a secondary free convective motion caused by buoyancy forces. Such behavior is in agreement with experimental observations.

Dimensional analysis of the pertinent equations reveals that the effect of free convection on the Sherwood number may be correlated approximately by the combined group $u_w Ra$. This is in addition to the usual film theory correction made to account for the finite interfacial velocity. Correlations for the asymptotic Sherwood number Sh_∞ and the pseudo effective transition length z_{tr} in terms of Rayleigh number Ra and dimensionless permeation velocity u_w for a system with constant product rate ($B_2 = 0$) and ideal membrane ($R_j = 1$) may be presented as

$$Sh_\infty = 1.09(u_w Ra)^{0.166}$$

$$z_{tr}^{1/3} = 1.19(u_w Ra)^{-0.166}, \quad \text{for} \quad u_w Ra > 7 \times 10^3$$

These expressions obtained over the range of $u_w = 0.01$, 0.1, 1, and $Ra \leqq 5 \times 10^7$ are in fair agreement with the experimental asymptotic Sherwood numbers of Derzansky and Gill (1974) and Hsieh et al. (1976). The analysis developed here may be used to predict the behavior of reverse osmosis systems, which are governed by the rejection parameter, Rayleigh number, and pressure parameter, in the combined forced and free convective region.

Reverse osmosis has been widely used in liquid phase separations such as desalination, pharmaceutical processing, and waste treatment. There is, in general, a buildup of a solute layer on the feed solution side of the semipermeable membrane which is designed to reject the solute component. The solute buildup, termed concentration polarization, establishes a solute concentration gradient near the phase boundary which increases the osmotic pressure π_w at the membrane surface and thus decreases the effective driving force ($\Delta P - \Delta\pi$) for the solvent flux. The effect of concentration polarization may become very significant as the quality of membrane, which is characterized by the rejection parameter R_j, increases. The extent of this effect has been studied theoretically and experimentally by many investigators (Dresner, 1964; Merten et al., 1964; Brian, 1965; Sherwood et al., 1965; Gill et al., 1966; Sherwood et al., 1967; Kimura and Sourirajan, 1968; Liu and Williams, 1970; Hendricks and Williams, 1971).

In addition to increasing the effective osmotic pressure, the concentration gradient due to polarization also induces a density variation in the fluid which has been neglected heretofore in most theories. Johnson and Acrivos (1969) included this density effect in the study of a two-dimensional vertical flat plate system. Ramanadhan and Gill (1969) theoretically found that natural convection due to density gradients had a significant effect on the reverse osmosis rate by applying a perturbation method to the system with vertical parallel plate geometry. In addition, Hendricks et al. (1972) and Johnson (1974) experimentally confirmed a buoyancy driven convection in reverse osmosis for an unstirred batch cell and a vertical flat plate, respectively. Further investigations of Derzansky and Gill (1974) lead to an empirical correlation for local Sherwood number having the form Sh_∞ = constant $\cdot Ra_D{}^{1/4}$ in the downstream region where free convection is important. Hsieh et al. (1976) continued the experiments of Derzansky and Gill and used a boundary layer integral analysis for free convection in a horizontal tubular flow. Their analysis simplified the complex flow fields from three dimensions to radial (R) and angular (ϕ) dependences only in the boundary layer by applying the assumption that the bulk concentration was equal to the feed concentration. This is a reasonably good assumption for very small leak off velocity. They found that the correlation proposed by Derzansky and Gill should be modified to $Sh_\infty = C_1 Ra_D{}^{1/4}$, where C_1 was related to the operating conditions. Their theory resulted in a maximum deviation of 32% from the experimental value of the local Sherwood number.

As was indicated by the review paper of Gill et al. (1971), reverse osmosis in a horizontal tubular flow involves three-dimensional convective diffusion and, of course, would be most correctly described by a three-dimensional model. It is thus the attempt of this work to develop a three-dimensional analysis for the horizontal reverse osmosis system with combined forced and free convection. In the mixed (that is, forced and free) convection region, such analyses have been made for heat transfer in horizontal tubes (Siegwarth et al., 1969; Shannon and Depew, 1969; Newell and Bergles, 1970; Cheng and Ou, 1974; Hong et al., 1974) and mass transfer in horizontal rectangular channels (Chang et al., 1976). Among these studies, the three-dimensional finite-difference approximations applied by Cheng and Ou, Hong et al., and Chang et al. generally are considered to be satisfactory. The previous analyses, however, are somewhat simpler than the corresponding reverse osmosis system because no net mass transfer occurs at the wall, and thus the interfacial velocity is zero. Hieber (1974) has made a boundary-layer perturbation analysis of the reverse osmosis problem; however, a direct comparison between his results and those presented here is not possible, since his work uses a uniform inlet velocity profile, while the current work employs a parabolic profile.

One difficulty in the reverse osmosis problem is caused by the mass flux through the wall, characterized by a leak off velocity (U_w), which prohibits using directly the stream function-vorticity scheme. However, this difficulty may be overcome by splitting the boundary conditions and governing equations into two parts, namely, a perturbation part which accounts for buoyancy effects when $Ra \neq 0$ and a nonperturbation part for $Ra = 0$. The permeation velocity U_w is associated with the nonperturbation equations, while the boundary conditions accounting for the solute flux are incorporated into the diffusion equation for solute in the perturbation equations. The nonperturbation equations for $Ra = 0$ thus become a purely fluid dynamics problem of internal flow with leak off, while the perturbation equations for $Ra \neq 0$ then can be cast into a stream function-vorticity framework.

Although the fluid dynamics problem with variable leak off velocity can be theoretically solved, a simple solution is available if U_w = constant (Berman, 1958; Yuan and Finkelstein, 1956). In reverse osmosis systems, the variation of U_w is usually very small (Gill et al., 1971). In this work, the solutions of Berman and Yuan and Finkelstein have been utilized together with circumferential and axial average values of U_w to approximate the velocity fields in the nonperturbation system. Having the nonperturbation solution, one can then proceed to solve the perturbation equations by appropriate methods.

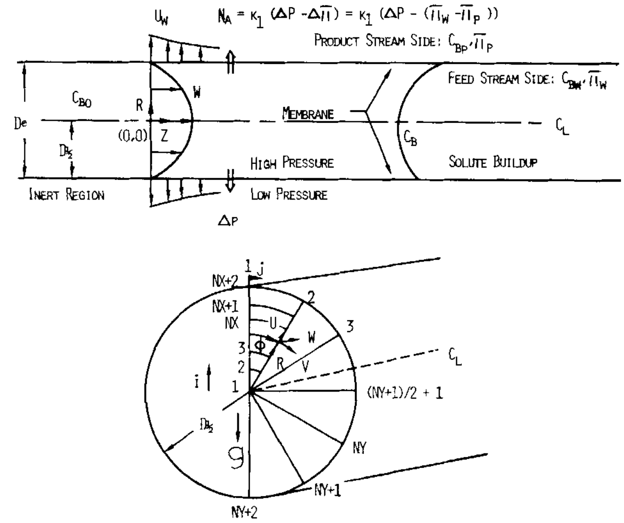

Fig. 1. Coordinate system and finite-difference grid for a tube.

Finite-difference approximations are employed to solve the governing perturbation equations. The numerical method essentially consists of an ADI scheme and an explicit method described as method V by Torrance (1968) which has desirable conservation properties and may be modified to be implicit in order to increase the step size.

MATHEMATICAL MODEL

A solution containing solvent A and solute B flows laminarly in a horizontal circular tube of radius $D_{1/2}$ as shown in Figure 1. The flow is hydrodynamically fully developed, and the tube has an inert inlet. A reverse osmosis process takes place at the pipe wall which is sealed with a semipermeable membrane which has characteristics enabling it to reject the solute. Generally, a solute layer, which is more concentrated than the core solution, builds up at the wall and results in a phenomenon termed concentration polarization. This polarization gives rise to a density difference in the fluid causing buoyancy forces which result in a secondary free convective motion superimposed upon the main axial flow. Since the density of a fluid in reverse osmosis is primarily a function of solute concentration, the usual Boussinesq approximation is applied by ignoring the density variation in all analytical expressions except in the body force term of the momentum equation. The complete three-dimensional equations of continuity and motion which govern the system and are valid for any Schmidt number are as follows (Bird et al., 1960*a*):

$$\nabla \cdot \mathbf{V} = 0 \tag{1}$$

$$\rho(\mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla P + \rho \mathbf{g} + \mu \nabla^2 \mathbf{V} \tag{2}$$

$$\mathbf{V} \cdot \nabla C_B = D_B \nabla^2 C_B \tag{3}$$

The applicable boundary conditions (B.C.) for (1) to (3) are the following:

at $Z = 0$

$$U = V = 0, \quad W = W_{oo}(R), \quad C_B = C_{B_0} \tag{4}$$

at $R = De/2$

$$U = U_w(C_{Bw}), \quad V = W = 0, \quad D_B \frac{\partial C_B}{\partial R} = R_j U_w C_B \tag{5}$$

In accord with the Boussinesq approximation, the variation of density is represented by the equation of state

$$\frac{\rho}{\rho_o} = 1 - \beta(C_B - C_{Bo}) \tag{6}$$

Introducing the perturbation quantities $\mathbf{V}'$ and P' by $\mathbf{V} = \mathbf{V}_O + \mathbf{V}'$ and $P = P_O + P'$, one may split the governing equations into the following two sets of equations. Nonperturbation equations:

$$\nabla \cdot \underline{V_O} = 0 \tag{7}$$

$$(\mathbf{V}_O \cdot \nabla)\mathbf{V}_O = \frac{-1}{\rho_o} \nabla P_O + \mathbf{g} + \nu \nabla^2 \mathbf{V}_O \tag{8}$$

with B.C.

at $Z = 0$

$$U_O = V_O = 0, \quad W_O = W_{OO}(R) \tag{9}$$

at $R = De/2$

$$U_O = U_w(C_{Bw}), \quad V_O = W_O = 0 \tag{10}$$

Perturbation equations accounting for the buoyancy effects:

$$\nabla \cdot \mathbf{V}' = 0 \tag{11}$$

$$(\mathbf{V}_O \cdot \nabla)\mathbf{V}' + (\mathbf{V}' \cdot \nabla)(\mathbf{V}_O + \mathbf{V}') =$$

$$\frac{-1}{\rho_o} \nabla P' = \left(\frac{\rho}{\rho_o} - 1\right)\mathbf{g} + \nu \nabla^2 \mathbf{V}' \tag{12}$$

$$(\mathbf{V}_O + \mathbf{V}') \cdot \nabla C_B = D_B \nabla^2 C_B \tag{13}$$

with B.C.

at $Z = 0$

$$U' = V' = W' = 0, \quad C_B = C_{Bo} \tag{14}$$

at $R = De/2$

$$U' = V' = W' = 0, \quad D_B \frac{\partial C_B}{\partial R} = R_j U_w C_B \tag{15}$$

For reverse osmosis, the leak off velocity, which may be represented in dimensionless form as

$$u_w = (k_1 \Delta P/\rho)(De/D_B)[1 - B_2 R_j(\theta_{Bw} + 1)] \tag{16}$$

decreases from the inlet to the downstream region; however, the variation usually is small (Gill et al., 1971). Therefore, for a system with small leak off rate, u_w may be regarded as a constant in order to obtain a reasonable solution for the system. The problem corresponding to (7) to (10) thus becomes a pipe flow with constant leak off velocity u_w. Berman (1953), in the solution for incompressible laminar flow in channels with porous walls, has shown that when u_w and hence the wall Reynolds number $Re_w(=u_w De/\nu)$ are small, the inertia terms in the equations of motion can be neglected. Thus, referring to Berman (1958) and Yuan and Finkelstein (1956), one may approximate the solution for small $u_w(<2\nu/D_B)$ in the dimensionless form:

$$v_o = 0$$

$$w_o = W_o/W_{oo} = 2(1 - 4r^2)(1 - 4\overline{u}_{ow,z} z) \tag{17}$$

$$u_o = 4r(1 - 2r^2)\overline{u}_{ow,\phi} \tag{18}$$

It is noted that θ_{Bw} varies with circumferential coordinate ϕ, and therefore a circumferential average value of u_{ow} denoted as $\overline{u}_{ow,\phi}$ is used to obtain u_o in (18). In order to account for the axial variation of leak off velocity, an axial average value of u_{ow} is applied in (17) to obtain a local value of w_o at any axial position.

Several workers (for example, Chang et al., 1976; Sorensen and Stewart, 1974; Ozoe and Churchill, 1972) have observed that allowing $Sc \to \infty$ and $Pe \to \infty$ give a reasonable approximation for this type of problem. The inertia terms for secondary flow in the momentum equations are negligible for large Sc, and the axial diffusion terms are not important for large Pe. This simplification reduces the perturbation equations to

$$\nabla^2 \xi = Ra\left(\frac{\partial \theta_B}{\partial r} \sin\phi + \frac{1}{r} \frac{\partial \theta_B}{\partial \phi} \cos\phi\right) \tag{19}$$

$$\nabla^2 \psi = \xi \tag{20}$$

$$\nabla^2 \theta_B = (u_o + u') \frac{\partial \theta_B}{\partial r} + \frac{v'}{r} \frac{\partial \theta_B}{\partial \phi} + w_o \frac{\partial \theta_B}{\partial z} \tag{21}$$

where one sets $v_o = 0$ and applies $w' = 0$ by noting (15). The two-dimensional Laplacian operator is defined as $\nabla^2 = \partial^2/\partial r^2 + \partial/r\partial r + \partial^2/r^2\partial\phi^2$. The appropriate B.C. for (19) to (21) are

at the pipe wall, $r = ½$

$$\psi = \partial\psi/\partial\phi = \partial\psi/\partial r = 0 \tag{22a}$$

$$\xi = \partial^2\psi/\partial r^2 \tag{22b}$$

$$\frac{\partial \theta_B}{\partial r} = R_j u_{ow}(\theta_B + 1) \tag{22c}$$

at the vertical center line, $\phi = 0, \pi$

$$\psi = \partial^2\psi/\partial\phi^2 = \partial\psi/\partial r = 0 \tag{23a}$$

$$\xi = 0 \tag{23b}$$

$$\frac{\partial \theta_B}{\partial \phi} = 0 \tag{23c}$$

The Ra number in Equation (19) appears naturally as a result of placing the equations in dimensionless form and is based on the constant inlet solute concentration.

QUANTITIES OF INTEREST

The local Sherwood number for solute B is of primary interest and can be obtained as

$$Sh^*_B = \overline{\frac{\partial \theta_{Bw}}{\partial r}} \Big/ (\overline{\theta}_{Bw} - \overline{\theta}_B) \tag{24}$$

where

$$\overline{\theta}_B = \iint_{A^*_c} w_o \theta_B dA^*_c \Big/ \iint_{A^*_c} w_o dA^*_c \tag{25}$$

$$\overline{\theta}_{Bw} = \frac{1}{S^*} \int_{S^*} \theta_{Bw} dS^* \tag{26}$$

$$\left(\overline{\frac{\partial \theta_{Bw}}{\partial r}}\right) = \frac{1}{S^*} \int_{S^*} \left(\frac{\partial \theta_{Bw}}{\partial r}\right) dS^* \tag{27}$$

$$dS^* = r d\phi$$

$$dA^*_c = r dr d\phi$$

In evaluating $(\overline{\partial \theta_{Bw}/\partial r})$, one may apply (22c) and (16). The concentration polarization, which induces the density gradient, may be defined as

$$\Gamma = (\overline{C}_{Bw} - \overline{C}_B)/\overline{C}_B = (\overline{\theta}_{Bw} - \overline{\theta}_B)/(\overline{\theta}_B + 1) \tag{28}$$

COMPUTATIONAL PROCEDURE

The inlet values of concentration θ_B on the boundaries, leak off velocity u_{ow}, radial velocity u_o, and axial velocity

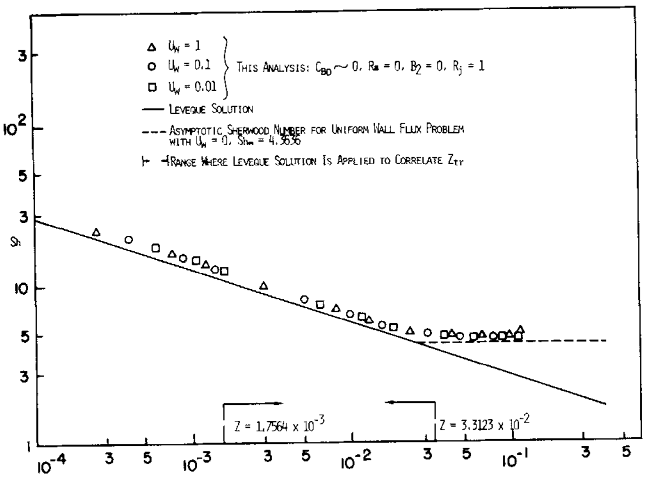

Fig. 2. Comparison of local Sherwood number for $Ra = 0$ with Leveque solution.

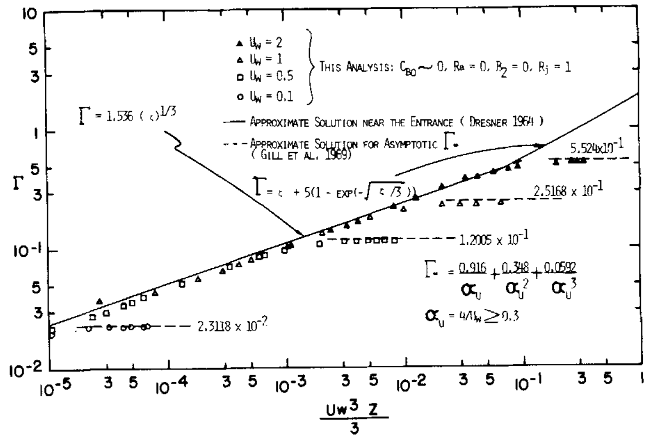

Fig. 3. Comparison of solutions for $Ra = 0$.

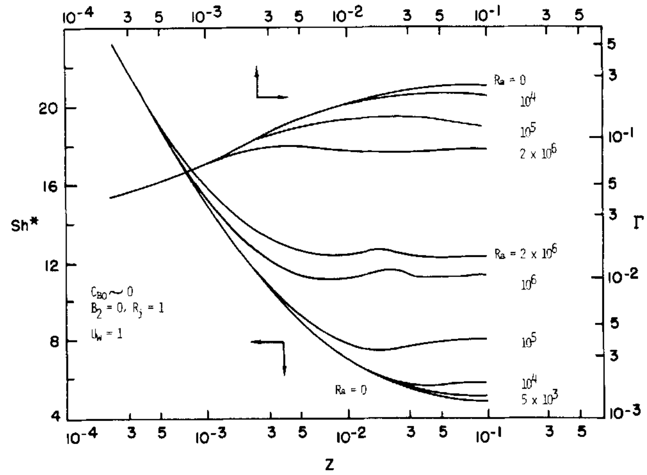

Fig. 4. Effect of Rayleigh number on local Sherwood number and concentration polarization.

w_o were first computed to initiate the solution. At any axial position, the interior concentration fields were found from the previously known field values and secondary velocity components at the prior axial location. Equation (21) was cast into a conservation form to insure the conservation properties of concentration field (Torrance, 1968; Torrance and Rockett, 1969). A modified Crank-Nicholson method was employed. Following this, the boundary concentration values u_{ow}, u_o, and w_o were calculated again at the new position. The stream function was obtained by introducing fictitious time steps into Equations (19) and (20) and applying the ADI method to solve the resulting equations until the steady stream function-vorticity fields were obtained and the boundary conditions for ξ_w were satisfied [Equation (22*b*)]. Evaluation of the secondary velocity components and computation of quantities of interest were then made. Further details of the analysis are given by Chang (1978).

TEST OF FINITE-DIFFERENCE METHOD

The computer program developed in this study was reduced and simplified to solve simpler problems with no leak off velocity; that is, $u_{ow} = 0$. The problems chosen to test the program were heat transfer in tubes with uniform wall heat flux given by Cheng and Ou (1974) and Hong et al. (1974) in which

$w_o = w_{oo}$ = constant for all z

$\partial\theta/\partial r = 2$, at the wall

$\overline{\theta} = 8z$

where

θ = dimensionless temperature, $(T - T_o)/\theta_c$

$\theta_c = D_{1/2}/q_{Tw}k_T$

q_{Tw} = uniform heat flux at wall

Test runs were made for $Ra = 0$, 8×10^4, 4×10^5 with several mesh sizes and various finite-difference approximations at the boundaries. Comparisons of Nu computed with a 11 × 11 grid gave agreement from within 5 to 10% of Hong et al. (1974) and Cheng and Ou (1974) for $Ra = 8 \times 10^4$. For $Ra = 0$, the numerical asymptotic Nu was 4.4 vs. the analytical value of 4.364. However, it should be mentioned that Cheng and Ou iterated the vorticity and stream function by using the previous ξ_w which appeared satisfactory, while Hong et al. applied a boundary vorticity method which assumed a linear relation between ξ_w and ψ_w. Based on the different approaches taken by these workers and the reasonable results obtained, it was determined that a satisfactory result was obtained by updating ξ_w values every twenty-

Table 1. Comparison of Sherwood Number (Sh^*) with Uniform Wall Flux Solution with Film Theory Correction (Sh^*_F) for $Ra = 0$, $B_2 = 0$, $R_j = 1$ with Different Leak off Velocities u_w

u_w	Sh^*_∞†	$Sh^*_{\infty F}$††
0.0001	4.453	4.3637
0.001	4.454	4.3641
0.01	4.460	4.3686
0.1	4.530	4.4138
0.5	4.817	4.6184
1.0	5.128	4.8827
2.0	5.707	5.4398
4.0	6.990	6.6650

† Sh^*_∞ = obtained by finite-difference program.

†† $Sh^*_{\infty F} = Sh_\infty \Theta_{AB} = Sh_\infty \left(\dfrac{-u_w/Sh_\infty}{\exp(-u_w/Sh_\infty) - 1} \right)$.

Sh_∞ = 4.3636, asymptotic Sherwood number for uniform wall flux problem.

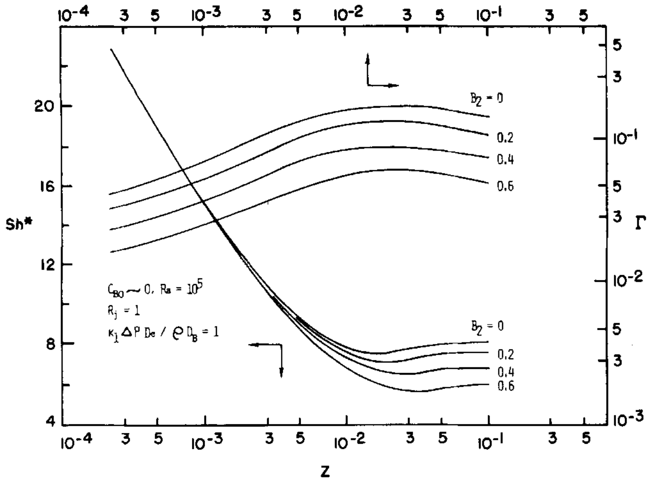

Fig. 5. Effect of pressure parameter on local Sherwood number and concentration polarization.

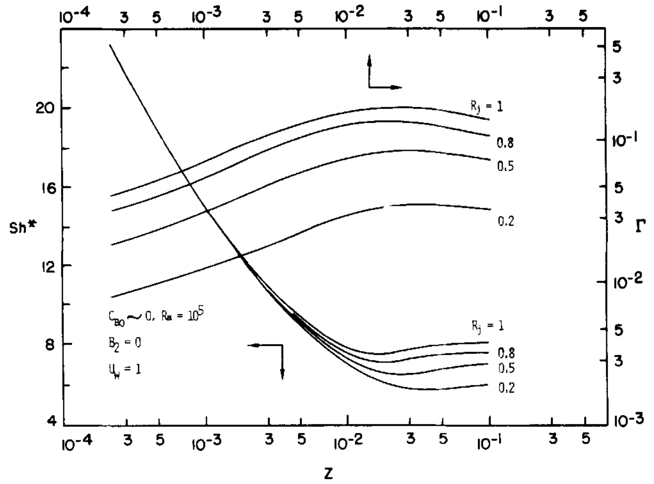

Fig. 6. Effect of rejection parameter on local Sherwood number and concentration polarization.

five steps. These comparisons also suggested the use of extrapolation to obtain boundary values of the concentration at the wall and vertical center line except at $r = 0$. These procedures resulted in a substantial reduction of computation time which was around 10 to 15 min on an IBM 370/158 with 11×11 grid.

Comparisons with analytical solutions for the reverse osmosis system were made for the case of $Ra = 0$. Film theory was used to account for the finite interfacial velocity (Bird et al., 1960*b*). The Sh^* and Sh were related according to

$$Sh^* = \theta_{AB} Sh \tag{29}$$

where

$$\theta_{AB} = \frac{-u_w/Sh}{\exp(-u_w/Sh) - 1} \tag{30}$$

or

$$Sh = \frac{-u_w}{\ln\left[1 - \left(\frac{u_w}{Sh^*}\right)\right]} \tag{31}$$

Figure 2 gives the local Sherwood number after converting from Sh^* to Sh with different leak off velocities. The results check very well with the analytical asymptotic value as well as with the theoretical Leveque solution for the uniform wall flux problem (Knudsen and Katz, 1958)

$$Sh_L = 1.301\, z^{-1/3} \tag{32}$$

which is valid and has been experimentally shown to be a good approximation for the reverse osmosis system near the entrance region (Derzansky and Gill, 1974; Hsieh et al., 1976). Additional comparison with the analytical solution for concentration polarization as presented in the next section further confirm the validity of the finite-difference method.

REVERSE OSMOSIS FOR $Ra = 0$

The effects of permeation velocity u_w on Sherwood number and concentration polarization are shown in Table 1 and Figure 3. It is seen that a larger value of u_w results in a larger amount of fluid leaking out from the system and hence increases the Sherwood number and concentration polarization. For a given u_w, increasing the concentration polarization would reduce the driving force and increase the resistance at the membrane. Both Sh^* and Γ approach constant values far downstream. Table 1 also compares asymptotic Sherwood numbers obtained by the finite-difference solution to analytical values from the uniform wall flux solution with film theory correction (Bird et al., 1960*b*). The concentration polarization shown in Figure 3 (also obtained from the finite-difference program) agrees well with the analytical solutions of Dresner (1964) and Gill et al. (1969). The asymptotes of Γ at various values of u_w are well predicted by the finite-difference solution.

EFFECT OF RAYLEIGH NUMBER

The variations of local Sherwood number and concentration polarization with Ra as parameter have been studied for various u_w. Typical results are presented in Figure 4 for $u_w = 1$. A buoyancy effect appears at a certain length z depending on the value of Ra and u_w. It has been found that for $u_w\, Ra > 7 \times 10^3$, free convection indeed becomes significant. As would be expected, a higher Ra induces greater mixing and yields a larger value of Sh^*. Near the entrance region, forced convection dominates and tends to decrease Sh^*; however, as z increases, the free convection contribution becomes more important and increases the Sh^*. Further downstream, both effects are important, and Sh^* approaches a nearly constant value. The increased mixing because of density effects also decreases the concentration polarization and thus increases the solvent flux across the membrane, as may be seen in Figure 4.

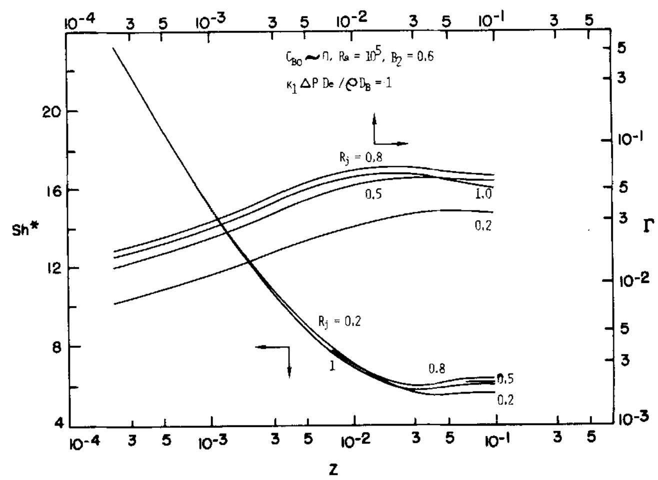

Fig. 7. Effect of rejection parameter on local Sherwood number and concentration polarization.

TABLE 2. ASYMPTOTIC SHERWOOD NUMBER (Sh^{*}_{∞}) AND PSEUDO EFFECTIVE TRANSITION LENGTH (z_{tr}) FOR $B_2 = 0$, $R_j = 1$, WITH DIFFERENT u_w AND Ra

	$u_w = 0.01$		$u_w = 0.1$		$u_w = 1$	
Ra	Sh^{*}_{∞}	z_{tr}	Sh^{*}_{∞}	z_{tr}	Sh^{*}_{∞}	z_{tr}
0	4.3686†		4.4138†		4.8827†	
10^3			4.591	7.18×10^{-2}	5.187	9.7039×10^{-2}
3×10^3					5.272	5.7766×10^{-2}
5×10^3					5.385	4.3611×10^{-2}
10^4	4.480	9.5965×10^{-2}	4.602	6.8466×10^{-2}	5.736	2.6872×10^{-2}
5×10^4			4.726	4.8818×10^{-2}		
10^5	4.496	8.3811×10^{-2}	4.999	3.2547×10^{-2}	8.110	6.5247×10^{-3}
5×10^5	4.632	5.2162×10^{-2}			10.42	3.0×10^{-3}
10^6	4.925	3.3123×10^{-2}	7.318	7.4626×10^{-3}	11.48	2.2462×10^{-3}
2×10^6			8.255	5.0×10^{-3}	12.54	1.7564×10^{-3}
5×10^6	6.374	1.1377×10^{-2}	9.560	3.4258×10^{-3}		
10^7	7.234	7.5792×10^{-3}	10.61	2.562×10^{-3}		
2×10^7	8.171	5.3×10^{-3}				
5×10^7	9.485	3.4588×10^{-3}				

† Uniform wall flux solution with film theory correction (Sh^{*}_{F}).

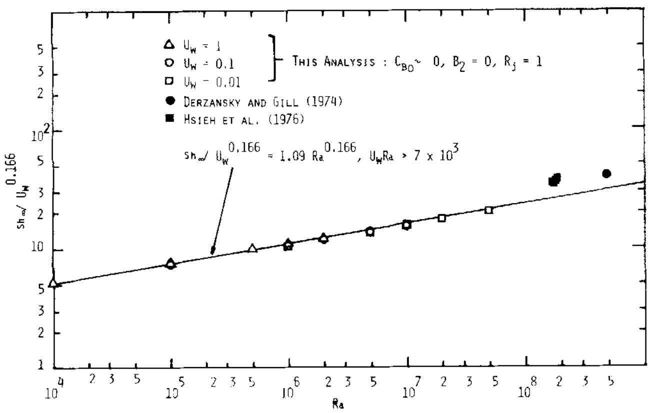

Fig. 8. Correlation for asymptotic Sherwood number.

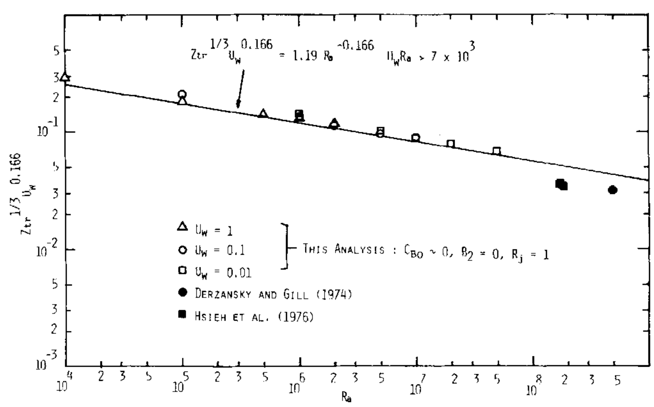

Fig. 9. Correlation for pseudo effective transition length.

EFFECT OF OPERATING PRESSURE

The effect of operating pressure parameter B_2 upon Sh^* and Γ is shown in Figure 5 for a perfect membrane ($R_j = 1$). As $B_2 (= \pi_o/\Delta P)$ increases, the permeation velocity u_w [Equation (16)] and hence $\partial\theta_{Bw}/\partial r$ [Equation (22c)] decrease. With less leakage of fluid through the membrane, the concentration polarization becomes smaller. The decreasing Sherwood number is mainly due to the small $\partial\theta_{Bw}/\partial r$ across the membrane. A similar conclusion may be drawn for a non-ideal membrane, for example, $R_j = 0.5$.

EFFECT OF REJECTION PARAMETER

Figure 6 shows the effect of membrane rejection parameter R_j upon Sh^* and Γ for the system with constant product rate ($B_2 = 0$). It is seen that the polarization and Sherwood number decrease as R_j decreases. The reason for the decreasing polarization level is clear if one notes that R_j is related to C_{Bp} and C_{Bw} as

$$R_j = 1 - \frac{C_{Bp}}{C_{Bw}}$$

A membrane with poor rejection quality results in concentrated C_{Bp} or dilute C_{Bw} and thus small polarization. The decreasing Sh^* is again caused by small $\partial\theta_{Bw}/\partial r$ due to small R_j. The permeation velocity u_w for $B_2 = 0$ is constant.

The axial variation of Sh^* and Γ for variable product rate, for example, $B_2 = 0.6$, is presented in Figure 7 with R_j as parameter. Examination of the results indicates that, in general, both Sh^* and Γ increase when R_j increases as in the case of $B_2 = 0$. However, Sh^* and Γ decrease at $R_j > 0.8$ and result in a crossover. This effect may be explained by noting that $\partial\theta_{Bw}/\partial r$ reaches a maximum for some value of R_j.

CORRELATION AND COMPARISON WITH EXPERIMENTAL DATA

The asymptotic Sherwood number Sh^{*}_{∞} for the system of constant product rate ($B_2 = 0$) and ideal membrane ($R_j = 1$) is presented in Table 2. These values may be reduced approximately to Sh values by using film theory according to Equation (31) and may be correlated as

$$Sh_{\infty} = 1.09(u_w Ra)^{0.166} \quad \text{for} \quad u_w Ra > 7 \times 10^3 \quad (33)$$

This correlation is noteworthy from two standpoints: first, because of the appearance of u_w, and second, because the power on Ra is not the usual one fourth. The appearance of u_w is not surprising, since, in addition to the usual effect of interfacial velocity on the mass transfer coefficient, the magnitude of u_w also affects the rate of boundary-layer development, and this latter effect cannot be accounted for by the film theory. The reason for this may be seen by introducing $\theta_B' = \theta_B/u_w$ into Equations (19) and (22c); then, for small values of θ_B and constant u_w, Equation (19) reveals that the free convective process is governed by the dimensionless group $u_w Ra$.

A comparison between the present analysis and experimental data of Derzansky and Gill (1974) is shown in Figure 8. Parameter values for their experiments were $De = 2.31 \times 10^{-2}$m, $\beta = 0.7$, $Sc = 529$, $C_{Bo} = 0.007818$, $(k_1 \Delta P De/\rho D_B) = 82.38$, $B_2 = 0.14$, $R_j = 0.91$, $u_w = 58.76$, $Ra = 4.883 \times 10^8$, and $Sh^*_\infty = 104$. For comparison purposes, the experimental Sh^*_∞ was converted to Sh_∞ using Equation (31). Comparison is also made with data of Hsieh et al. (1976) in the same figure ($B_2 = 0.073$, $R_j = 0.84$). The agreement with both sets of experimental data is satisfactory. One note of caution is appropriate with regard to the correlation of Figure 8 and Equation (33), this being that numerical results were obtained only up to $u_w Ra = 5 \times 10^7$, which is below the range of experimental data. Thus, extension of the results to values of $u_w Ra$ above this range represents an extrapolation and should be viewed accordingly. The deviations between the prediction and experiment for Sh^*_∞ are about 18% for the data of Derzansky and Gill (1974) and about 11 to 14% with those of Hsieh et al. (1976). In addition, the correlation developed here is for a perfect membrane and constant leak off rate ($R_j = 1$ and $B_2 = 0$). Although the numerical problem was solved for other cases, no correlation was attempted because of the increased number of parameters. However, approximate corrections for Sherwood numbers in the case of $B_2 \neq 0$, $R_j \neq 1$ may be made by using Figure 5 and 6 for B_2 and R_j, respectively. This procedure was followed in plotting the experimental data.

Table 2 also presents a pseudo effective transition length z_{tr} which is obtained by extrapolating Sh^*_∞ for $Ra > 0$ until it intersects the Sherwood number curve for $Ra = 0$, the position of intersection being termed $z = z_{tr}$. This transition length z_{tr} is the length at which free convection begins to have an effect on the reverse osmosis process. Since z_{tr} is near the entrance region, the Leveque solution for uniform wall flux may be used to represent the solution for $Ra = 0$ (Derzansky and Gill, 1974). The justification for this is shown in Figure 2. Using Equation (32), one has

$$z_{tr}^{1/3} = 1.19(u_w Ra)^{-0.166}, \quad \text{for} \quad u_w Ra > 7 \times 10^3 \tag{34}$$

The correlation for z_{tr} by (34) is shown in Figure 9 along with the present analysis and experimental data of Derzansky and Gill (1974) and Hsieh et al. (1976).

NOTATION

Ac = cross-section area of the pipe
ADI = alternating direction implicit
B_2 = fraction of applied pressure required to overcome the osmotic pressure of the feed solution, an operating parameter, $\pi_o/\Delta P$
C = mass fraction
C_L = center line
D = diffusivity
$D_{1/2}$ = radius of pipe, $De/2$
$d_{1/2}$ = dimensionless radius of pipe, ½
De = equivalent diameter of pipe
Gr = Grashof number, $\beta g De^3 \theta c/\nu^2$
g = gravitational acceleration
$\bar{h}$ = circumferential average heat transfer coefficient
k_L = mass transfer coefficient
k_T = thermal conductivity
k_1 = membrane coefficient for diffusion of solvent
N = mass flux
Nu = Nusselt number, $\bar{h}De/k_T$
NX, NY = number of interior grid lines in R, ϕ directions, respectively
P = pressure
Pc = a pressure term, $\rho_o \nu D_B/De^2$
ΔP = difference in applied pressure across the membrane, an operating parameter
Pe = Peclet number, $ScRe(=\overline{W}_{oo}De/D_B)$
P = dimensionless pressure, P/Pc
q_T = heat flux
R, Z = cylindrical coordinates in radial and axial directions, respectively
Ra = Rayleigh number, $GrSc(=\beta g De^3 \theta_c/\nu D_B)$
Ra_D = Rayleigh number based on concentration difference, $\beta g De^3(C_{Bw} - C_{Bo})/\nu D_B$
Re = Reynolds number, $\overline{W}_{oo}De/\nu$
R_j = rejection parameter for membrane, $1 - (C_{Bp}/C_{Bw})$
r, z = dimensionless cylindrical coordinates, R/De, $Z/(DePe)$
S = perimeter of pipe
Sc = Schmidt number, ν/D_B
Sh^* = Sherwood number, $\overline{k}_L{}^* De/D_B$
Sh = Sh^* with film theory correction, Equation (31)
T = temperature
t = fictitious time in ADI method (iteration parameter)
U, V, W = velocity components in r, ϕ, z directions
u, v, w = dimensionless velocity components, UDe/D_B, VDe/D_B, $W/\overline{W}_{oo}$
$\overline{u}_{ow,z}$ = dimensionless axial average value of U_{ow}, $\frac{1}{z}\int_0^z \overline{u}_{ow,\phi} dz$
$\overline{u}_{ow,\phi}$ = dimensionless circumferential average value of U_{ow}, $\frac{1}{2\pi d_{1/2}}\int_0^{2\pi} u_{ow} d_{1/2} d\phi$
W_{oo} = axial velocity for rectilinear flow at inlet, Poiseuille velocity component
w_{oo} = dimensionless velocity for W_{oo}, $2(1 - 4r^2)$
z_{tr} = dimensionless transition length

Greek Letters

α_u = $4/u_w$
β = volume expansion coefficient,

$$\frac{1}{-\rho_o}\frac{\partial \rho}{(\partial C_B)_o} = \frac{-(\rho - \rho_o)}{\rho_o(C_B - C_{Bo})}$$

Γ = concentration polarization,

$$(\overline{C}_{Bw} - \overline{C}_B)/\overline{C}_B\{=(\overline{\theta}_{Bw} - \overline{\theta}_B)/(\overline{\theta}_B + 1\}$$

$$\zeta = \frac{3}{8}\left(\frac{u_w{}^3 z}{3}\right)$$

θ_{AB} = correction factor for interfacial velocity
θ_B = dimensionless concentration of B, $(C_B - C_{Bo})/\theta_c$
θ_c = characteristic concentration for reverse osmosis system, C_{Bo}
μ = viscosity
ν = kinematic viscosity, μ/ρ
ξ = dimensionless vorticity,

$$\nabla^2\psi\left(=\partial^2\psi/\partial r^2 + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}\right)$$

π = osmotic pressure
$\Delta\pi$ = osmotic pressure difference across the membrane,

$$\pi_w - \pi_p \left(= \pi_w \left(1 - \frac{\pi_p}{\pi_w}\right) = \pi_o \frac{C_{Bw}}{C_{Bo}} \left(1 - \frac{C_{Bp}}{C_{Bw}}\right) = \pi_o \frac{C_{Bw}}{C_{Bo} R_j}\right)$$

ρ = density
ϕ = cylindrical coordinates in angular direction
ψ = dimensionless stream function,

$$u' = \frac{1}{-r}\frac{\partial\psi}{\partial\phi}, \quad v' = \partial\psi/\partial r$$

Subscripts

A, B = species A of solvent, B of solute
i, j = space subscripts of grid points in r and ϕ directions
o = inlet value at $z = 0$ or value in main nonperturbation flow
p = fluid condition adjacent to membrane surface of product stream side
w = wall evaluated quantity or fluid conditions adjacent to membrane surface of feed stream side
∞ = asymptotic value

Superscripts

$*$ = dimensionless variable or quantity with interfacial velocity effect
$-$ = average value
$'$ = perturbation quantity

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